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Coulomb wave function corrections for n-particle Bose–Einstein correlations

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Abstract. The effect of multi-particle Coulomb final state interactions on higher-order intensity correlations is determined in general, based on a scattering wave function which is a solution of the n-body Coulomb Schrödinger equation in (a large part of) the asymptotic region of n-body configuration space. In particular, we study Coulomb effects on the n-particle Bose–Einstein correlation functions of similarly charged particles and remove a systematic error as big as 100% from higher-order multi-particle Bose–Einstein correlation functions.

1 Introduction

One of the most fundamental quests of high-energy physics is the determination of the phase diagram of strongly interacting matter. At high densities and/or temperatures the quarks are expected to be liberated from their confinement within hadrons and a new phase of matter, the quark gluon plasma (QGP), to be formed. In order to explore this new phase of matter, the Relativistic Heavy Ion Collider (RHIC) has been constructed at Brookhaven National Laboratory, to collide Au + Au nuclei at $s^{1/2} = 200$ AGeV center of mass energy.

One of the new features of RHIC physics will be the production of 600–1200 charged pions per unit rapidity. Due to this, the PHENIX and STAR detectors will be able to determine not only single particle spectra and the 2-particle Bose–Einstein correlations, but also the higher-order Bose–Einstein correlation functions, which turn out to be essential even up to 5th order, to be able to distinguish experimentally the fully chaotic and the partially coherent particle sources [1–3]. As partial coherence is a fundamental aspect of quantum fields, and it can be related to a possible Bose–Einstein condensation of pion wave-packets to the wave-packet with the smallest energy in the rest frame of the source [4], the determination of the higher-order Bose–Einstein correlation functions at RHIC is of great theoretical interest [1–3, 5–9], as well as a great experimental challenge.

However, Coulomb (and possibly strong) final state interactions of the pions play an important rôle in shaping the final multi-particle Bose–Einstein correlation functions. As no consistent and systematic treatment of the final state interaction of a charged multi-boson system is available in the literature, the experimental removal of the Coulomb effects from the n-particle Bose–Einstein correlation functions is based hitherto only on some ad hoc generalization of the Gamow formula to the multi-particle case.

In this paper, we propose a straightforward method for a *systematic* quantum-mechanical treatment of Coulomb final state interactions in higher-order Bose–Einstein correlation functions. Although the validity of the method described below is limited to a certain, albeit large, kinematic domain ($\Omega_0^{(n)}$) due to the fact that the exact solution of even the 3-body Coulomb scattering problem is beyond presently available means, we think that the results presented here represent a first important step towards establishing a link between few-body physics and Bose–Einstein correlations in high-energy multi-particle physics. Especially, the new Coulomb wave function corrections indicate that the generalized Gamow correction method would make a factor of two error in the 5th order Bose–Einstein correlation functions, if the radius parameters were in the 5–10 fm range, as is characteristic for heavy ion collisions. If the characteristic radius parameters were as small as 1 fm, the characteristic size in reactions of high-energy particle physics, the generalized Gamow factors would be acceptable, at a 10% level of precision in the 5th order Bose–Einstein correlation functions.

The strength of higher-order correlation functions increases much slower for partially coherent particle sources than for incoherent sources with an unresolvable halo of long-lived resonances [2, 1]. Even if the strength of the 2nd and 3rd order Bose–Einstein correlations were similar in a partially coherent and another incoherent particle source, the strength of the 5th order correlation functions would be a factor of 2 different between the partially coherent and the fully chaotic cases [1]. In order to distinguish these scenarios, the Coulomb final state interactions must be corrected for, and the error on the Coulomb correction must be kept under control.

In a recent Letter, we have presented a refined treatment of the 3-body Coulomb correction problem [11], with application to new high-energy heavy ion data by the NA44 experiment [12]. In the present work, we generalize this 3-body Coulomb wave function integration method to the case of n-particle Coulomb corrections.

2 Bose–Einstein n-particle correlations and final state interactions

Let us summarize some properties of the Bose–Einstein n-particle correlation functions using only the generic aspects of their derivation, and establish a link between the theory of final state interactions in few body physics and the theory of Bose–Einstein correlations in high-energy particle and nuclear physics.

The n-particle Bose–Einstein correlation function is defined as

$$C_n(\mathbf{k}_1, \cdots, \mathbf{k}_n) = \frac{N_n(\mathbf{k}_1, \cdots, \mathbf{k}_n)}{N_1(\mathbf{k}_1) \cdots N_1(\mathbf{k}_n)}, \quad (1)$$

where $N_n(\mathbf{k}_1, \cdots, \mathbf{k}_n)$ is the n-particle inclusive invariant momentum distribution, while $N_1(\mathbf{k}_1)$ is the single particle invariant momentum distribution. It is quite remarkable that this complicated object, which carries quantum-mechanical information on the phase-space distribution of particle production as well as on the possible partial coherence of the source, can be expressed in a relatively simple, straightforward manner both in the analytically solvable pion-laser model of [13,4] as well as in the generic boosted-current formalism of Gyulassy and Padula [14] as

$$C_n(\mathbf{k}_1, \cdots, \mathbf{k}_n) = \frac{\sum_{\sigma^{(n)}} \prod_{i=1}^{n} G(\mathbf{k}_i, \mathbf{k}_{\sigma_i})}{\prod_{i=1}^{n} G(\mathbf{k}_i, \mathbf{k}_i)}, \quad (2)$$

where $\sigma^{(n)}$ stands for the set of permutations of indices $(1, 2, \cdots, n)$ and σ_i denotes the element replacing element i in a given permutation from the set of $\sigma^{(n)}$, and, regardless of the details of the two different derivations,

$$G(\mathbf{k}_i, \mathbf{k}_j) = \langle a^\dagger(\mathbf{k}_i) a(\mathbf{k}_j) \rangle \quad (3)$$

stands for the expectation value of $a^\dagger(\mathbf{k}_i) a(\mathbf{k}_j)$. In the boosted-current formalism, the derivation is based on the assumptions that

(i) the bosons are emitted from a semi-classical source, where currents are strong enough so that the recoils due to radiation can be neglected,
(ii) the particle sources are an incoherent random ensemble of such currents, described by a boost-invariant formulation [14], and
(iii) that the particles propagate as free plane waves after production.

However, a formally similar result is obtained when particle production happens in a correlated manner, and even final state interactions between the produced particles are allowed for, generalizing the results of [4,15,16].

In the pion-laser model, the n-particle exclusive invariant momentum distributions read

$$N_n^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n) = \sum_{\sigma^{(n)}} \prod_{i=1}^{n} G_1(\mathbf{k}_i, \mathbf{k}_{\sigma_i}), \quad (4)$$

with

$$G_1(\mathbf{k}_i \mathbf{k}_j) = \mathrm{Tr}\{\hat{\rho}_1 a^\dagger(\mathbf{k}_i) a(\mathbf{k}_j)\}, \quad (5)$$

where $\hat{\rho}_1$ is the single particle density matrix in the limit when higher-order Bose–Einstein correlations are negligible. One can show [15,9], that the n-particle inclusive spectrum has a similar structure, if the multiplicity distribution is Poissonian in the rare gas limit:

$$N_n(\mathbf{k}_1, \cdots, \mathbf{k}_n) = \sum_{\sigma^{(n)}} \prod_{i=1}^{n} G(\mathbf{k}_i, \mathbf{k}_{\sigma_i}), \quad (6)$$

$$G(\mathbf{k}_i, \mathbf{k}_j) = \sum_{n=1}^{\infty} G_n(\mathbf{k}_i, \mathbf{k}_j). \quad (7)$$

The functions $G_n(\mathbf{k}_i, \mathbf{k}_j)$ can be considered as representatives of order n symmetrization effects in exclusive events; see [13,4,15] for more detailed definitions. The function $G(\mathbf{k}_i, \mathbf{k}_j)$ can be considered as the expectation value of $a^\dagger(\mathbf{k}_i) a(\mathbf{k}_j)$ in an inclusive sample of events, and this building block includes all the higher-order symmetrization effects. In the relativistic Wigner-function formalism, in the plane wave approximation $G(\mathbf{k}_1, \mathbf{k}_2)$ can be rewritten as

$$G(\mathbf{k}_1, \mathbf{k}_2) = \int \mathrm{d}^4 x S(x, K_{12}) \exp(\mathrm{i} q_{12} \cdot x), \quad (8)$$

$$K_{12} = 0.5(k_1 + k_2), \quad (9)$$

$$q_{12} = k_1 - k_2, \quad (10)$$

where a four-vector notation is introduced, and $a \cdot b$ stands for the inner product of four-vectors, and $k = ((m^2 + \mathbf{k}^2)^{1/2}, \mathbf{k})$. Due to the mass-shell constraints, i.e. $E_{\mathbf{k}} = (m^2 + \mathbf{k}^2)^{1/2}$, G depends only on six independent momentum components. In any given frame, the boost-invariant decomposition of (10) can be rewritten into the following seemingly non-invariant form:

$$G(\mathbf{k}_1, \mathbf{k}_2) = \int \mathrm{d}^3 \mathbf{x} S_{\mathbf{K}_{12}}(\mathbf{x}) \exp(\mathrm{i} \mathbf{q}_{12} \mathbf{x}), \quad (11)$$

$$S_{\mathbf{K}_{12}}(\mathbf{x}) = \int \mathrm{d}t \exp(\mathrm{i} \boldsymbol{\beta}_{K_{12}} \mathbf{q}_{12} t) S(\mathbf{x}, t, K_{12}), \quad (12)$$

$$\boldsymbol{\beta}_{K_{12}} = (\mathbf{k}_1 + \mathbf{k}_2)/(E_1 + E_2). \quad (13)$$

Note that the relative source function $S_{\mathbf{K}_{12}}(\mathbf{x})$ reduces to a simple time integral over the source function $S(x, K)$ in the frame where the mean momentum of the pair (hence the pair velocity $\boldsymbol{\beta}_{K_{12}}$) vanishes.

If n particles are emitted with similar momenta, so that their n-particle Bose–Einstein correlation functions may be non-trivial, (5) and (7) will form the basis for the evaluation of the Coulomb and strong final state interaction effects on the observables. On this level, all the correlations are build up from correlations of pairs of particles. This is due to the specific form of the density matrix that includes just the right amount of stimulated emission to make a further calculation straightforward. Note also that a similar result can be obtained in the semi-classical boosted-current formalism, where the particle production has negligible effect on the elementary source of pion production (bremsstrahlung-like radiation).

Let us point out that the exact solution of multi-particle Bose–Einstein symmetrization in the pion-laser model resulted in a Poisson cluster picture [15,4]. This implies that in the rare gas limit, without Coulomb or other final state interactions, the multi-boson correlations appear only as a random admixture of a small amount of correlated pairs to independently distributed single particles. As the density increases, also the fraction of correlated pairs increases and the admixture of independently distributed clusters of particle triplets, quartets, and higher-order n-tuples becomes correspondingly more important. The result of [15,4] indicates that below the onset of Bose–Einstein condensation, a fully symmetrized multi-boson system can be considered as a convolution of independently distributed clusters of particle n-tuples, and it is natural to apply Coulomb corrections within such clusters of particles only. We shall also discuss that, when one of the particles becomes separated from its cluster, the relevant n-particle Coulomb correction factor will reduce to the Coulomb correction factor of a smaller cluster that contains the remaining $n-1$ particles.

3 Quantum-mechanical treatment of the Coulomb n-body problem

In order to treat correctly the Coulomb corrections to the n-particle correlation function, knowledge of the n-body Coulomb scattering wave function is required. We restrict ourselves to the case that the transverse momenta of all the particles in the final state in their center of mass are small enough to make a non-relativistic approach sensible. Hence the problem consists of finding the solution of the n-charged particle Schrödinger equation when all n particles are in the continuum.

Consider n distinguishable particles with masses m_i and charges $e_i, i = 1, 2, \cdots, n$. Let $\mathbf{x}_i$ and $\mathbf{k}_i$ denote the coordinate and momentum (three-)vectors, respectively, of particle i. From these we construct in the usual manner the relative coordinate $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ and the relative momentum $\mathbf{k}_{ij} = (m_j\mathbf{k}_i - m_i\mathbf{k}_j)/(m_i + m_j)$ between particles i and j, the corresponding reduced mass being $\mu_{ij} = m_i m_j/(m_i + m_j)$.

The n-particle Schrödinger equation reads

$$\left\{ H_0 + \sum_{i<j=1}^{n} V_{ij} - E \right\} \Psi^{(+)}_{\mathbf{k}_1\cdots\mathbf{k}_n}(\mathbf{x}_1, \cdots, \mathbf{x}_n) = 0, \quad (14)$$

where

$$E = \sum_{i=1}^{n} \frac{\mathbf{k}_i^2}{2m_i} > 0 \quad (15)$$

is the total kinetic energy for n particles in the continuum. H_0 is the free Hamilton operator and

$$V_{ij}(\mathbf{r}_{ij}) = V^{\mathrm{S}}_{ij}(\mathbf{r}_{ij}) + V^{\mathrm{C}}_{ij}(\mathbf{r}_{ij}) \quad (16)$$

the interaction potential between particles i and j, consisting of a strong but short-range (V^{S}_{ij}) plus the long-range Coulomb interaction ($V^{\mathrm{C}}_{ij}(\mathbf{r}_{ij}) = e_i e_j / \mid \mathbf{r}_{ij} \mid$). Equation (14) has to be complemented by the complete set of boundary conditions in order to obtain a unique solution.

Already for $n = 3$, the exact numerical solution of the Schrödinger equation (14) for $E > 0$ is beyond present means, partly for principal and partly for practical reasons. For a brief discussion of the related difficulties see [17]. But, at least the complete set of boundary conditions to be imposed is nowadays known analytically [18, 19], in the form of the explicit solutions of the Schrödinger equation in all asymptotic regions of the 3-particle configuration space. Apart from the trivial 2-cluster region relevant for an asymptotic configuration containing only two particles one of which is a bound state of two particles, the asymptotic solution takes its simplest form in the asymptotic region conventionally denoted by Ω_0 and characterized by the fact that – roughly speaking – all three interparticle distances become uniformly large, i.e. all $\mid \mathbf{r}_{ij} \mid\to \infty$ (for a precise definition of the various asymptotic regions see [19]). There exist three more asymptotic regions $\Omega_{ij}, i < j = 1, 2, 3$, which are pertinent to situations characterized by final state interactions between particles i and j. But the appropriate asymptotic solutions are rather more complicated. From the physical point of view the union of all these regions $\Omega_0 \bigcup \Omega_{12} \bigcup \Omega_{13} \bigcup \Omega_{23}$ is relevant for the complete break-up into three free particles. As has been shown in [11], in spite of the lack of an exact solution of the 3-body Schrödinger equation in the whole 3-body configuration space, already knowledge of the asymptotic solution in Ω_0 led to a systematic, well-controlled extraction of Coulomb effects in the 3-particle Bose–Einstein correlation measurements, in contrast to earlier, ad hoc 3-body Coulomb correction methods.

In the final states of heavy ion reactions, where a large number of charged particle tracks appear, the mutual macroscopically large separation of tracks is one of the criteria of a clean measurement. This suggests that in order to study n-body correlation functions, again knowledge of the wave function in $\Omega_0^{(n)}$, the region in n-particle configuration space where all interparticle distances become uniformly large, i.e., $\mid \mathbf{r}_{ij} \mid\to \infty$ for all values of (ij), may be sufficient. Here, uniform divergence of interparticle distances in $\Omega_0^{(n)}$ means roughly that $0 <\mid \mathbf{r}_{ij} \mid / \mid \mathbf{r}_{kl} \mid< \infty$ for asymptotically large times for any two arbitrarily chosen particle pairs, although the interparticle distances themselves diverge for any pair. One immediate consequence of this definition is that in $\Omega_0^{(n)}$ the short-range interaction parts V_{ij}^{S} play no rôle any longer and can thus be neglected.

For want of an exact n-particle Coulomb scattering wave function an approximate solution of (14) is sought. For this purpose, let us introduce the continuum solution of the 2-body Coulomb Schrödinger equation by

$$\left\{-\frac{\Delta_{\mathbf{r}_{ij}}}{2\mu_{ij}}+V_{ij}^{\mathrm{C}}(\mathbf{r}_{ij})-\frac{\mathbf{k}_{ij}^2}{2\mu_{ij}}\right\}\psi_{\mathbf{k}_{ij}}^{\mathrm{C}(+)}(\mathbf{r}_{ij})=0, \quad (17)$$

describing the relative motion of the two particles i and j with energy $\mathbf{k}_{ij}^2/2\mu_{ij}$. The explicit solution is

$$\psi_{\mathbf{k}_{ij}}^{\mathrm{C}(+)}(\mathbf{r}_{ij}) = N_{ij}\ \mathrm{e}^{\mathrm{i}\mathbf{k}_{ij}\mathbf{r}_{ij}} \times F[-\mathrm{i}\eta_{ij},1;\mathrm{i}(\mid \mathbf{k}_{ij}\mid\mid \mathbf{r}_{ij}\mid - \mathbf{k}_{ij}\mathbf{r}_{ij})], \quad (18)$$

with $N_{ij} = \mathrm{e}^{-\pi\eta_{ij}/2}\Gamma(1+\mathrm{i}\eta_{ij})$, and $\eta_{ij} = e_i e_j \mu_{ij}/\mid \mathbf{r}_{ij}\mid$ being the appropriate Coulomb parameter. $F[a,b;x]$ is the confluent hypergeometric function and $\Gamma(x)$ the Gamma function. Thus the following ansatz for an approximate n-particle Coulomb wave function is made:

$$\Psi_{\mathbf{k}_1,\cdots,\mathbf{k}_n}^{(+)}(\mathbf{x}_1,\cdots,\mathbf{x}_n) \sim \prod_{i<j=1}^{n}\psi_{\mathbf{k}_{ij}}^{\mathrm{C}(+)}(\mathbf{r}_{ij}). \quad (19)$$

This ansatz can be justified by the following arguments.

(i) The wave function (19) is asymptotically correct in the asymptotic region $\Omega_0^{(n)}$; that is, it is the leading term if all interparticle separations go to infinity of the (unknown) exact solution of the Schrödinger equation (14) [18]. Of course, for non-asymptotic particle separations it represents a theoretically not compulsory though plausible extrapolation.

(ii) In the formal, time-dependent scattering theory the basic object is the MØLLER operator which maps the free n-particle state onto the corresponding scattering state. The mathematically rigorous definition of the n-charged particle MØLLER operator [20] requires, in contrast to the case of purely short-range interactions between the particles, the introduction of a 'renormalization factor'. The latter has the form of a product of $n(n-1)/2$ renormalization factors each of which is appropriate for the definition of the MØLLER operator for one of the possible pairings of the charged particles. Obviously, the ansatz (19) of the n-particle wave function is consistent with this renormalization prescription.

(iii) By suitably decomposing the (stationary) MØLLER operator of an n-body system into a chain of MØLLER operators of subsystems with fewer interacting particles, a wave function of the type (19) has been suggested as a lowest-order term of an n-particle Coulomb wave function in [21] to be used in all of configuration space. The assumptions entering were neglect of genuine higher-than-two particle correlations in the wave function, which is justified in $\Omega_0^{(n)}$, and restriction of all 2-particle scatterings onto their respective energy shells.

(iv) The wave function (19) coincides for any selected particle triplet with the form pertinent to the given preselected triplet, if the corresponding interparticle distances diverge [11].

(v) For $n=3$, such an approximate wave function has been proposed in [23,24]. Although it ceases to be a solution of the Schrödinger equation (14) for non-asymptotic values of the relative coordinates, it is nevertheless widely used, with considerable success, to calculate cross sections for the ionization of hydrogen atoms by the impact of an (energetic) electron.

The foregoing discussion makes it clear that the ansatz (19) is justifiable only for sufficiently large interparticle separations, a condition which is not easily translated into an experimentally accessible criterion. However, for $n=3$ such a criterion has been established, namely that the total kinetic energy $E_{\mathrm{total}}^{(3)}$ of three particles of unit charge be at least $0.2\hbar c/R_G(\mathrm{fm})$ which equals 10 MeV for a typical source size $R_G = 4\,\mathrm{fm}$ [11]. Thus, assuming also for an arbitrary number n of particles (again for simplicity taken to have unit charge) the total kinetic energy $E_{\mathrm{total}}^{(n)}$ to be equally distributed over the relative kinetic energies between each pair, the latter criterion generalizes to the condition $E_{\mathrm{total}}^{(n)} \geq 0.033n(n-1)\hbar c/R_G(\mathrm{fm})\,\mathrm{MeV}$. Hence, for the following we *assume* the n-particle Coulomb wave function to be given everywhere as

$$\begin{aligned}\Psi_{\mathbf{k}_1\cdots\mathbf{k}_n}^{(+)}(\mathbf{x}_1,\cdots,\mathbf{x}_n) &\approx \tilde{\Psi}_{\mathbf{k}_1\cdots\mathbf{k}_n}^{(+)}(\mathbf{x}_1,\cdots,\mathbf{x}_n),\\ &:= \sqrt{\mathcal{N}^{(n)}}\prod_{i<j=1}^{n}\psi_{\mathbf{k}_{ij}}^{\mathrm{C}(+)}(\mathbf{r}_{ij}),\\ \text{for } E_{\mathrm{total}}^{(n)} &\geq 0.066\frac{n(n-1)}{2}\frac{\hbar c}{R_G(\mathrm{fm})}\ \mathrm{MeV}, \quad (20)\end{aligned}$$

where $\mathcal{N}^{(n)}$ is an undetermined overall normalization constant. This is the building block for a properly symmetrized n-body wave function where the bosonic or fermionic nature of any subset of identical particles has to be taken into account in the symmetrization (or anti-symmetrization) process in the standard manner [22].

In this paper we explicitly present the fully symmetrized wave function only for the case of n identical charged bosons, as to our knowledge measurements in high-energy physics attempting to reveal the strength of the multi-particle Bose–Einstein correlation effects [12, 10] exist only for this special case.

The fully symmetrized n-particle wave function has the form

$$\tilde{\Psi}_{\mathbf{k}_1\cdots\mathbf{k}_n}^{(+)\mathcal{S}}(\mathbf{x}_1,\cdots,\mathbf{x}_n) = \frac{1}{\sqrt{n!}}\sum_{\sigma^{(n)}}\tilde{\Psi}_{\mathbf{k}_1\cdots\mathbf{k}_n}^{(+)}(\mathbf{x}_{\sigma_1},\cdots,\mathbf{x}_{\sigma_n}), \quad (21)$$

where $\sigma^{(n)}$ stands for the set of permutations of n different indices, and σ_i for the permuted value of the index i in one of the permutations belonging to the set $\sigma^{(n)}$. Using the ansatz (20), the above equation simplifies to

$$\tilde{\Psi}^{(+)\mathcal{S}}_{\mathbf{k}_1\cdots\mathbf{k}_n}(\mathbf{x}_1,\cdots,\mathbf{x}_n) = \frac{\sqrt{\mathcal{N}^{(n)}}}{\sqrt{n!}}\sum_{\sigma^{(n)}}\prod_{i<j=1}^{n}\psi^{\mathrm{C}(+)}_{\mathbf{k}_{ij}}(\mathbf{r}_{\sigma_i\sigma_j}), \quad (22)$$

which contains only the 2-body relative Coulomb wave functions.

The physics of the above ansatz is very simple: if all n final charges emerge into the continuum and if all are well separated from the other tracks, only the pairwise Coulomb relative wave functions play a rôle. However, relative Coulomb wave functions have to be taken into account for all possible particle pairs as the Coulomb interaction is of long range. Graphically, if we represent the n particles by n crosses, the relative Coulomb wave function between particle i and j can be represented by a line connecting cross i with cross j, and the full, asymptotically correct n-particle Coulomb wave function is represented by connecting each of the n crosses with the $n-1$ others by forming a polygon with n corners and $n(n-1)/2$ lines (diagonals and edges).

One can apply a simple approximation to (22) which preserves at least some features of the Coulomb distortion effects. It consists in neglecting, for any 2-particle Coulomb wave function $\psi^{\mathrm{C}(+)}_{\mathbf{k}_{ij}}(\mathbf{r}_{ij})$, the hypergeometric function in the exact solution (18) and retaining only the part $\mathrm{e}^{\mathrm{i}\mathbf{k}_{ij}\mathbf{r}_{ij}}N_{ij}$. After evaluating the double sums over all permutations of σ_n in a product, one finds

$$\left|\tilde{\Psi}^{(+)\mathcal{S}}_{\mathbf{k}_1\cdots\mathbf{k}_n}(\mathbf{x}_1,\cdots,\mathbf{x}_n)\right|^2 = \frac{\mathcal{N}^{(n)}}{n!}\left(\prod_{i<j=1}^{n}G_{ij}\right)\left|\sum_{\sigma^{(n)}}\prod_{i<j=1}^{n}\mathrm{e}^{\mathrm{i}\mathbf{k}_{ij}\mathbf{r}_{\sigma_i\sigma_j}}\right|^2 = \frac{\mathcal{N}^{(n)}}{\mathcal{N}_0}\left(\prod_{i<j=1}^{n}G_{ij}\right)\left|\Psi^{(0)\mathcal{S}}_{\mathbf{k}_1\cdots\mathbf{k}_n}(\mathbf{x}_1,\cdots,\mathbf{x}_n)\right|^2. \quad (23)$$

In the last line, the symmetrized n-particle wave function for neutral particles $\Psi^{(0)\mathcal{S}}_{\mathbf{k}_1\cdots\mathbf{k}_n}(\mathbf{x}_1,\cdots,\mathbf{x}_n)$ with its own normalization constant $\mathcal{N}_0$ has been introduced. As usual,

$$G_{ij} := |N_{ij}|^2 = \mathrm{e}^{-\pi\eta_{ij}}\,|\Gamma(1+\mathrm{i}\eta_{ij})|^2, \quad (24)$$

is the Gamow penetration factor for the particle pair (ij).

It thus follows that the proper generalization of the Gamow penetration factor for n-charged particles reads

$$G_{1,\cdots,n} = \prod_{i<j=1}^{n}G_{ij}. \quad (25)$$

This expression contains $n(n-1)/2$ factors, corresponding to all possible pairings (ij). Moreover, it is self-consistent: if the momenta $k', k'', \cdots, k^{(l)}$ of l particles approach infinity such that for no two momenta $k^{(i)}$ and $k^{(j)}$ the corresponding relative momentum remains finite, we have

$$\lim_{k'\to\infty}\cdots\lim_{k^{(l)}\to\infty}G_{1,\cdots,n} = G_{\alpha_1,\cdots,\alpha_{n-l}}, \quad (26)$$

where the remaining $n-l$ particles whose momenta remain finite are denoted symbolically by $\alpha_1,\cdots,\alpha_{n-l}$. Specifically,

$$\lim_{k_n\to\infty}G_{1,\cdots,n} = G_{1,\cdots,(n-1)}. \quad (27)$$

An explicit check for $n=4$ shows that, indeed,

$$\lim_{k_4\to\infty}G_{1,2,3,4} = \lim_{k_4\to\infty}G_{12}G_{13}G_{14}G_{23}G_{24}G_{34}, = G_{12}G_{13}G_{23} = G_{1,2,3}. \quad (28)$$

Hence, the generalization (25) of the Gamow correction factor to arbitrary values of n is done in a self-consistent manner that satisfies its physically expected reduction property.

This result was substantiated for the case of n identical charged bosons. In general, the final state of a high-energy heavy ion reaction contains many different kind of particles, with different charges and quantum statistical properties. Nevertheless, the ansatz given in (20) can be utilized for any values of the charges, and the result can be symmetrized for a generic mixture of particles as prescribed in [22].

Let us add two comments.

(i) As mentioned above, a wave function of the type (19) implies that the relative motion of each of the pairs of particles is independent of that of the other pairs, i.e., that no correlations between the motions of the particle pairs occur. In other words, the proposed form of the factorized n-particle Coulomb wave function does not include genuine higher-order correlations, only those that can be built up from 2-particle Coulomb correlations. This is the same level of approximation that is used to derive (2), the generic form of the n-particle Bose–Einstein correlation functions. Results of [4,15] suggest that (2) is valid only if the density of bosons is below the limit of Bose–Einstein condensation.

(ii) It should be kept in mind that the extrapolation out of the region $\Omega_0^{(n)}$ implied by (19) is highly non-unique. Even for the 3-body Coulomb problem, various different wave functions which, of course, coincide asymptotically in $\Omega_0^{(3)}$ with (19) have been, and are still being, developed.

4 Application to high-energy heavy ion and particle collisions

The correlation function measuring the enhanced probability for emission of n identical Bose particles is given by (1). This correlation function is usually, due to meager statistics, only measured as a function of the Lorentz invariant Q_n defined by the relation

$$Q_n^2 = \sum_{i<j=1}^{n}q_{ij}^2, \quad (29)$$

where $q_{ij} = k_i - k_j$, and where k_i is the four-momentum of particle i.

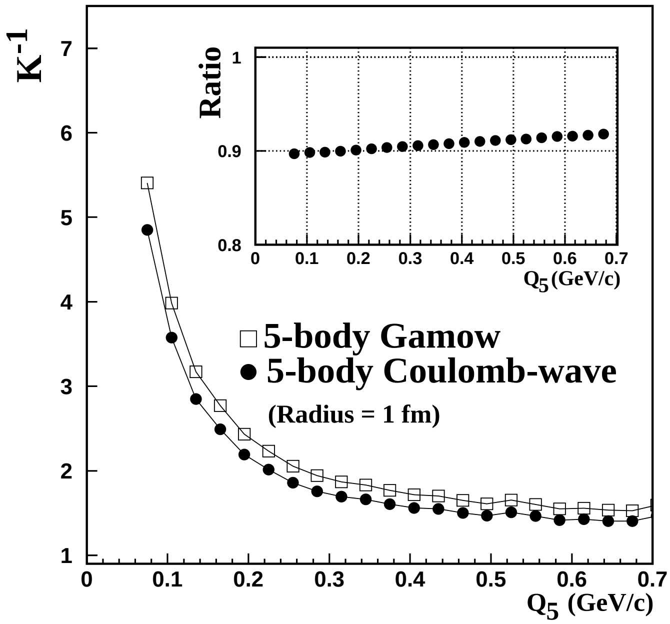

Fig. 1. Filled circles stand for the Coulomb correction factor of 5-particle Bose–Einstein correlation functions for a source size of $R = 1$ fm as obtained from the numerical integration of the 5-body Coulomb wave function, while the squares indicate the results of the less substantiated 5-body Gamow corrections; the inset shows the ratio of these two correction factors. Lines are shown to guide the eye

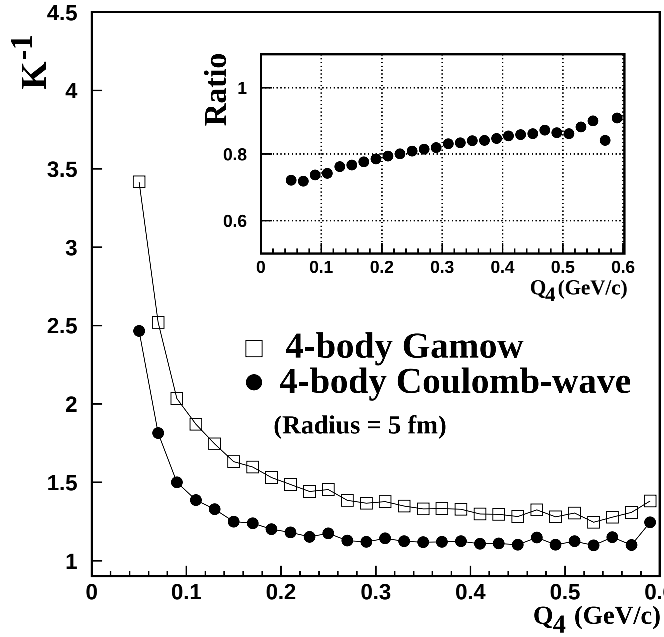

Fig. 2. Same as Fig. 1 but for $n = 4$ and $R = 5$ fm

We can now calculate the Coulomb effects on the n-particle correlation function using

$$K_{\mathrm{Coulomb}}(Q_n) = \frac{\int \prod_{i=1}^{n} \mathrm{d}^3\mathbf{x_i}\rho(\mathbf{x_i}) \left| \tilde{\Psi}^{(+)\mathcal{S}}_{\mathbf{k}_1\cdots\mathbf{k}_n}(\mathbf{x}_1,\cdots,\mathbf{x}_n)\right|^2}{\int \prod_{i=1}^{n} \mathrm{d}^3\mathbf{x_i}\rho(\mathbf{x_i}) \left| \Psi^{(0)\mathcal{S}}_{\mathbf{k}_1\cdots\mathbf{k}_n}(\mathbf{x}_1,\cdots,\mathbf{x}_n)\right|^2}, \quad (30)$$

where $\rho(\mathbf{x_i})$ is the density distribution of the source for particle i, taken as a Gaussian distribution of width R in all three spatial directions. This formulation makes it possible to extract information on the source size R, and to compare this value with that extracted by means of a generalized n-particle Gamow approximation through $K^{(G)}_{\mathrm{Coulomb}}(Q_n) = \prod_{i<j=1}^{n} G_{ij}$. To this purpose we use the NA44 data sample of three pion events produced in S–Pb collisions at CERN [12].

We have calculated the Coulomb correction factor, i.e. $K^{-1}_{\mathrm{Coulomb}}(Q_n)$ [12], for source radius values $R = 1$, 5, and 10 fm, for $n = 2$, 3, 4, and 5 particle correlations. The radii were chosen to be in the range of interest for high-energy particle and high-energy heavy ion physics. The results are compared to the generalized Gamow approximation. We have checked that in the limit $R \to 0$ the n-particle Gamow approximation is indeed recovered numerically.

In case of a characteristic 1 fm effective source size, typical for Bose–Einstein correlation functions in various elementary particle reactions, the difference between the n-particle Gamow and Coulomb wave function corrections were smaller than 10% for $n = 4$ and 5 particles, the $n = 5$ case being shown in Fig. 1. However, for future measurements of 5-particle Bose–Einstein correlations in particle physics that aim at a precision better than 5% relative error, Coulomb wave function integration will be a necessity.

For source sizes of 5 or 10 fm, that are the characteristic expectations for Au + Au reactions at RHIC, the difference between the results of the Gamow and Coulomb wave function corrections increased dramatically, see Figs. 2–4. We find that for a source radius of 5 fm, we need to take this detailed calculation into account already for the precise determination of the 3-particle correlation function. However, with increasing number of particles, the deviation between the n-particle Gamow and the n-particle Coulomb wave function integration method increases drastically. For 5-particle Coulomb correction, the better substantiated Coulomb wave function integration method yields a deviation factor of 2 from the naive generalized Gamow method. Finally, we note that systematic improvements of our treatment are possible by

(i) including also effects of strong interactions between the particles of each pair,
(ii) replacing the simple product of Gaussians by a more realistic model for the production of particle n-tuples, and
(iii) invoking improved n-body Coulomb wave functions that are correct in a larger region of n-body configuration space than $\Omega_0^{(n)}$.

However, corrections (i) and (ii) are estimated to be small, as

(1) the final state Coulomb interaction dominates over the final state strong interaction due to its long range and the relatively large source size, and

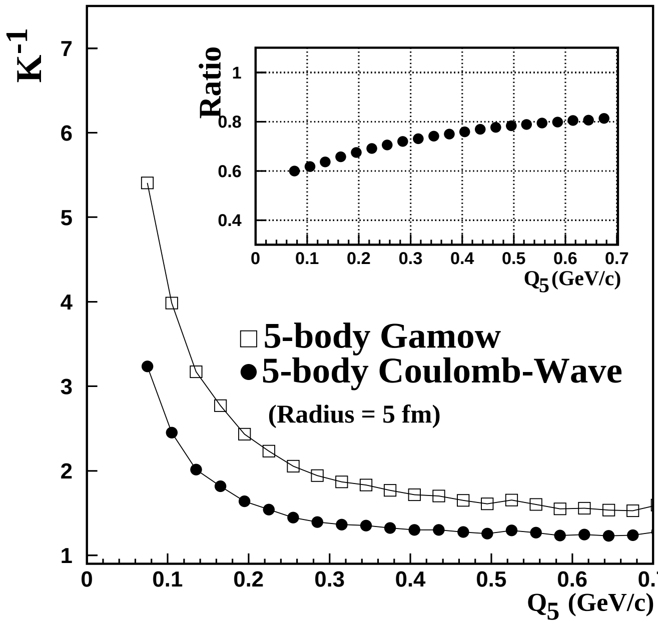

Fig. 3. Same as Fig. 1 but for $n = 5$ and $R = 5\,\text{fm}$

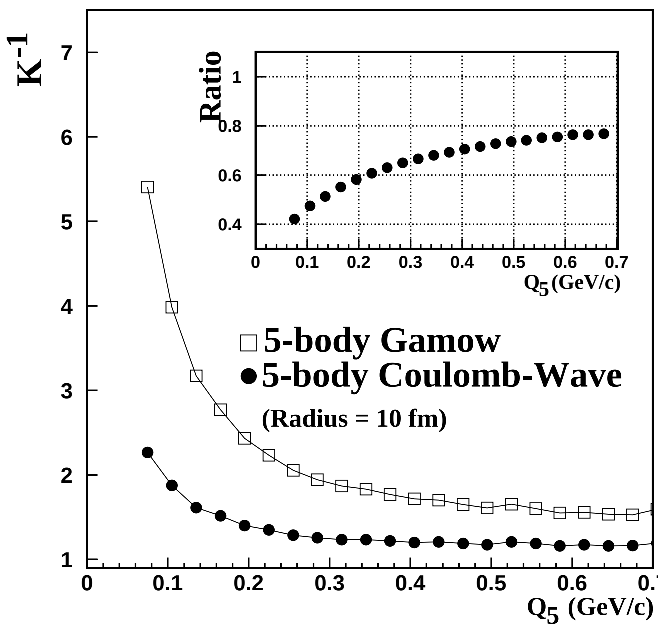

Fig. 4. Same as Fig. 1 but for $n = 5$ and $R = 10\,\text{fm}$

(2) the effective source of particles is known to be approximately Gaussian from detailed studies of two pion correlations.

Correction (iii) is also expected to be small, as a clean measurement of particle n-tuples will likely require that these particles be in $\Omega_0^{(n)}$; however, one has to wait till 4th and 5th order correlations are measured in heavy ion collisions in order to determine the more detailed experimental conditions.

5 Summary and conclusions

On the basis of an explicit, analytically given form of the n-body Coulomb wave function that is – at least asymptotically – correct in a large region of n-body configuration space, we have developed a new method to systematically correct for *explicit* many-body Coulomb effects which is applicable to data analysis in a broad range of measurements in high-energy particle and heavy ion physics. A generalized Gamow correction factor has been established as a limiting case of vanishing source sizes.

Specifically, we have worked out our approach for 3, 4, and 5 identical charged particles and have tested it for Gaussian source sizes with $R = 1$, 5, and 10 fm. We have numerically found that the generalized Gamow approximation is not reliable enough to determine the magnitude on the 5% level of the 5-body Coulomb correction factor if $R = 1\,\text{fm}$, the characteristic length scale of strong interactions in high-energy particle physics. The range of interest in high-energy heavy ion physics was probed in the $R = 5$ and 10 fm cases, and systematic errors, as large as 100%, were shown to be generated with the earlier Coulomb correction techniques for the correlation function of five particles.

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